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S^1 ACTIONS ON 6-MANIFOLDS†

LLOYD LININGER

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THERE have been many interesting results obtained concerning the classification of free differentiable actions of a compact Lie group G on a differentiable manifold M (e.g. [2], [3], [4], [6] and [17]). In general these results have shown that G cannot act differentiably on M or have classified the actions on some particular manifold M. In this paper we will classify all free differentiable actions of S^1 on 1-connected 6-manifolds. The results are more complicated to work with when $H_2 M$ has torsion, although the basic ideas are the same. We will therefore assume that $H_2 M$ is free until the last section of the paper, where we will discuss the classification when $H_2 M$ has torsion. The following theorem is the main result when $H_2 M$ is free.

THEOREM 4. For each positive integer k, there exist exactly two 1-connected 6-manifolds M_1 and M_2 such that, each of M_1 and M_2 admits a free differentiable S^1 action, $H_2 M_i$ is free and the rank of $H_2 M_i$ is k for $i = 1$ and 2. Furthermore, one of M_1 and M_2 is

$$S^2 \times S^4 \# \cdots \# S^2 \times S^4 \# S^3 \times S^3 \# \cdots \# S^3 \times S^3,$$

say M_1, and admits two distinct free S^1 actions. The manifold M_2 is not stably parallelizable and admits only one free S^1 action.

This result together with the results of [2] which classified the actions of S^1 on 2-connected 6-manifolds completes the classification of free differentiable S^1 actions on 1-connected 6-manifolds except for a more explicit determination of the manifold M_2 above. It is known that these manifolds admit uncountably many distinct free topological circle actions [5].

§1. CLASSIFICATION OF THE ORBIT SPACES

We will suppose throughout this section that M is a 1-connected 6-manifold, that $H_2 M$ is free with rank k where k is a positive integer, and that G is a differentiable free action of S^1 on M.

We will let M/G denote the orbit space of the action, K will denote the non-trivial 3-sphere bundle over S^2, and L will denote $S^2 \times S^3$.

THEOREM 1. The manifold M/G is either the connected sum of $(k + 1)$-copies of L or the connected sum of K and k-copies of L.

Proof. Looking at the exact homotopy sequence of the fibration $S^1 \to M \to M/G$, we see that $\pi_1(M/G) = 0$. From the Gysin sequence of the fibration we see that $H_2(M/G)$ is free and

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has rank $k + 1$. In [1, Theorem 2.3], Barden shows that $L \# \cdots \# L$, $(k + 1)$-copies, and $K \# L \# \cdots \# L$, k-copies of L, are the only differentiable manifolds satisfying these conditions, which are torsion free.

§2. CLASSIFICATION OF THE CIRCLE ACTIONS

We will next determine that there is a unique manifold and action which has the connected sum of $(k + 1)$-copies of L as its orbit space.

THEOREM 2. *The manifold* $S^2 \times S^4 \# \cdots \# S^2 \times S^4 \# S^3 \times S^3 \# \cdots \# S^3 \times S^3$, *with* k-*copies of* $S^2 \times S^4$ *and* $k + 1$-*copies of* $S^3 \times S^3$ *admits a free differentiable action of* S^1 *with* $L \# \cdots \# L$, $k + 1$-*copies, as its orbit space; there is only one such action on this manifold, and this is the only* 1-*connected* 6-*manifold admitting a free* S^1 *action with* $L \# \cdots \# L$ *as its orbit space.*

Proof. We will first show that there is precisely one 6-manifold with this property. Suppose M_1 and M_2 are 1-connected 6-manifolds with free differentiable S^1 actions G_1 and G_2 such that M_1/G_1 and M_2/G_2 are diffeomorphic to $L \# \cdots \# L$. Then G_1 and G_2 determine principle S^1 bundles over $L \# \cdots \# L$ and each bundle is classified by a homotopy class of maps from $L \# \cdots \# L$ into CP^∞, the classifying space for principle S^1 bundles. Hence we have the following commutative diagrams

$$\begin{tikzcd} M_1 \arrow[r, "F"] \arrow[d] & S^\infty \arrow[d] \\ M_1/G_1 \arrow[r, "f"] & CP^\infty \end{tikzcd} \qquad \begin{tikzcd} M_2 \arrow[r, "H"] \arrow[d] & S^\infty \arrow[d] \\ M_2/G_2 \arrow[r, "h"] & CP^\infty. \end{tikzcd}$$

Since CP^∞ is a $K(Z, 2)$, the homotopy classes of maps from M_i/G_i into CP^∞ are classified by the image under the induced map of the generators of $H_2(M_i/G_i; Z)$. Now by looking at the Gysin sequences of the fibrations associated with G_i and of the universal bundle, it is easy to see that for suitably chosen bases, the maps f^* and h^* either map a generator of $H_2(M_i/G_i; Z)$ to a generator of $H_2(CP^\infty; Z)$ or to zero. Now since M_i is simply connected it follows that at least one of the generators of $H_2(M_i/G_i)$ is mapped to a generator of $H_2(CP^\infty)$. Let $a_1, \ldots, a_{k+1}$ and $b_1, \ldots, b_{k+1}$ denote generators of $H_2(M_1/G_1)$ and $H_2(M_2/G_2)$ respectively. We can assume that $f^*(a_t)$ and $g^*(b_s)$ is the same generator for $1 \leq t \leq r$ and $1 \leq s \leq u$. Now there exists an isomorphism α' from $H_2(M_1/G_1)$ to $H_2(M_2/G_2)$ such that for each i, $h^* \alpha'(a_i) = f^*(a_i)$. In [1], Barden proved that there exists a diffeomorphism α from M_1/G_1 to M_2/G_2 which induces α'. Then the diagram

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d] \arrow[rr, bend left, "F"] & M_2 \arrow[r, "H"] \arrow[d] & S^\infty \arrow[d] \\ M_1/G_1 \arrow[r, "\alpha"] \arrow[rr, bend right, "f"'] & M_2/G_2 \arrow[r, "h"] & CP_\infty \end{tikzcd}$$

commutes up to homotopy. It follws that α is covered by a bundle map which defines a diffeomorphism from M_1 to M_2 which proves the actions G_1 and G_2 are equivalent.

It should be observed that by changing the basis in a suitable way we could have assumed that $f^*(a_i)$ was a generator iff $i = 1$. Similarly for h^*.

For the existence of such an action we can define a map from $L\# \cdots \#L$ into CP^∞ which takes the natural generator of $H_2(L)$, from the first factor, into the generator of $H_2(CP^\infty)$ and all other generators to zero. This is an easy obstruction theory argument. Then the induced total space of this bundle is the 6-manifold

$$M = [S^3 \times S^3 - (\text{int}\,\{S^1 \times D^5\})] \cup [S^1 \times (L\# \cdots \# L) - (\text{int}\,\{S^1 \times D^5\})]$$

with a suitable identification on the boundary $S^1 \times S^4$. (The last factor has k-copies of L.) Now we wish to show that M is $S^2 \times S^4 \# \cdots \# S^2 \times S^4 \# S^3 \times S^3 \# \cdots \# S^3 \times S^3$. Again using the Gysin sequence we know that M is 1-connected and rank $H_2 M = k$. By Wall [8], it follows that $M = N^6 \# S^3 \times S^3 \# \cdots \# S^3 \times S^3$ with $(k+1)$-copies of $S^3 \times S^3$, since the Euler characteristic of M is zero. It also follows from [8], that N^6 will be the connected sum of k-copies of $S^2 \times S^4$ if the first Pontryagin class of M is zero.

The manifolds $S^3 \times S^3$ and $S^1 \times (L \# \cdots \# L)$ are parallelizable, and if P denotes $S^3 \times S^3 - (\text{int}\,\{S^1 \times D^5\})$ and Q denotes $S^1 \times (L \# \cdots \# L) - (\text{int}\,\{S^1 \times D^5\})$, then P and Q are parallelizable. Now $p_1[M] \in H^4(M; Z)$ and by looking at the Mayer–Vietoris sequence $\to H^3(S^1 \times S^4; Z) \to H^4(M; Z) \xrightarrow{i_P^* \oplus i_Q^*} H^4(P; Z) \to H^4(Q; Z) \to$, we see that $(i_P^* \oplus i_Q^*)(p_1[M])$ is zero. This follows from the fact that P and Q are submanifolds of M of dimension 6 and the Pontryagin classes are natural. But $H^3(S^1 \times S^4; Z)$ is zero so $p_1[M]$ must be zero. This concludes the proof of Theorem 2.

THEOREM 3. *There are two 1-connected 6-manifolds which admit a free differentiable circle action with orbit space $K\# L\# \cdots \# L$, k-copies of L. One of these 6-manifolds is $S^2 \times S^4 \# \cdots \# S^2 \times S^4 \# S^3 \times S^3 \# \cdots \# S^3 \times S^3$.*

Proof. We want to study the homotopy classes of maps from $K \# L \# \cdots \# L$ into CP^∞ with the property that the total space of the induced bundle is 1-connected. This is equivalent to assuming that the induced map is non-trivial on $H_2(K \# L \# \cdots \# L)$. We will let f denote a map from $K \# L \# \cdots \# L$ into CP^∞ such that $f_* \mid H_2(K \# L \# \cdots \# L)$ is non-trivial. It follows from the Gysin sequence that there is a basis $\beta_1, \ldots, \beta_{k+1}$ for $H_2(K\# L \# \cdots \# L)$ such that $f_*(\beta_i)$ is the same generator of $H_2(CP^\infty)$ for each i. Now $K \# L \# \cdots \# L$ has non-zero 2nd Stiefel–Whitney class since K has non-zero 2nd Steifel–Whitney class. Recalling that for simply-connected manifolds the second Steifel–Whitney class can be thought of as a homomorphism from the second homology of the manifold with integral coefficients to Z_2, we can assume that $\omega_2(\beta_1) \neq 0$.

We will now define two circle actions with orbit space $K \# L \# \cdots \# L$. Since

$$H_2(K \# L \# \cdots \# L; Z) \approx H_2(K; Z) \oplus \cdots \oplus H_2(L; Z),$$

let $\gamma_1, \ldots, \gamma_{k+1}$ be a basis for $H_2(K \# L \# \cdots \# L)$ where each γ_i is a generator of the ith factor in the above sum. Then $\omega_2(\gamma_1) \neq 0$ and $\omega_2(\gamma_2) = \omega_2(\gamma_3) = \cdots = \omega_2(\gamma_{k+1}) = 0$. Let h and g be maps from $K \# L \# \cdots \# L$ into CP^∞ such that $h_*(\gamma_1)$ is a generator of $H_2(CP^\infty)$,

$g_*(\gamma_2) = h_*(\gamma_1)$, h_* takes $\gamma_2, \gamma_3, \ldots, \gamma_{k+1}$ to zero, and g_* takes $\gamma_1, \gamma_3, \gamma_4, \ldots, \gamma_{k+1}$ to zero. An easy obstruction theory argument shows that such maps exist. First we show that any action with orbit space $K \# L \# \cdots \# L$ is equivalent to one of the actions induced by these two circle bundles. Then we will classify these two actions.

Step 1. There exists a basis $\alpha_1, \ldots, \alpha_n$ for $H_2(K \# L \# \cdots \# L)$ such that $\omega_2(\alpha_1) \neq 0$, $\omega_2(\alpha_i) = 0$ for $i > 1$, and $f_*(\alpha_i)$ is either a generator or zero.

Proof. Let $\alpha_1 = \beta_1$. For $i > 1$, let $\alpha_i = \beta_i$ when $\omega_2(\beta_i) = 0$ and $\alpha_i = \beta_i - \alpha_1$ when $\omega_2(\beta_i) \neq 0$. It is clear that $\alpha_1, \ldots, \alpha_n$ has the desired properties.

Step 2. Suppose $f_*(\alpha_1)$ is a generator of $H_2(CP^\infty)$ and $f_*(\alpha_i)$ is zero for $i > 1$. Then the action induced by f is equivalent to the action induced by h above.

Proof. This follows immediately from [1] since the map $\alpha_i \to \pm \gamma_i$ is an isomorphism preserving ω_2 and hence is induced by a diffeomorphism from $K \# L \# \cdots \# L$ onto itself. This diffeomorphism will be covered by a bundle map which gives the equivalence of actions.

Step 3. Suppose $f_*(\alpha_i)$ is a generator for some $i > 1$. Then the action induced by f is equivalent to the action induced by g above.

Proof. The following map from $H_2(K \# L \# \cdots \# L)$ into itself is an isomorphism. Let $\alpha_1 \to \gamma_1$ if $f_*(\alpha_1) = 0$ or $\alpha_1 \to \gamma_1 - \gamma_2$ if $f_*(\alpha_1)$ is a generator. Let $\alpha_2 \to \gamma_i$ if $f_*(\alpha_2) = 0$ or $\alpha_2 \to \gamma_2 - \gamma_i$ if $f_*(\alpha_2)$ is a generator. Let $\alpha_i \to \gamma_2$. For all other j, let $\alpha_j \to \gamma_j$ if $f_*(\alpha_j) = 0$ or $\alpha_j \to \gamma_2 - \gamma_j$ if $f(_*\alpha_j)$ is a generator. Then as in step 2, this isomorphism preserves ω_2 and is induced by some diffeomorphism. This diffeomorphism is again covered by a bundle map which gives the equivalence of the actions.

Step 4. The actions induced by the maps h and g are distinct and the total space of the circle bundle induced by h is

$$S^2 \times S^4 \# \cdots \# S^2 \times S^4 \# S^3 \times S^3 \# \cdots \# S^3 \times S^3.$$

Proof. To see that the total space of the circle bundle induced by h is

$$S^2 \times S^4 \# \cdots \# S^2 \times S^4 \# S^3 \times S^3 \# \cdots \# S^3 \times S^3$$

we argue as in the proof of Theorem 2. To see that h and g are distinct we observe that there is an embedded 2-sphere in $K \# L \# \cdots \# L$ which represents γ_1 and hence has a non-trivial normal bundle. Since $g_*(\gamma_1)$ is zero, this 2-sphere lifts to a 2-sphere in the total space of the induced bundle with non-trivial normal bundle. Therefore, the induced total space has non-zero second Stiefel–Whitney class and is not homotopy equivalent to the induced total space of h.

THEOREM 4. *For each positive integer k, there exist exactly two 1-connected 6-manifolds M_1 and M_2 such that, each of M_1 and M_2 admits a free differentiable S^1 action, and $H_2(M_i)$ is free and has rank k for $i = 1$ and 2. Furthermore, one of M_1 and M_2 is*

$$S^2 \times S^4 \# \cdots \# S^2 \times S^4 \# S^3 \times S^3 \# \cdots \# S^3 \times S^3,$$

say M_1, and admits two distinct free S^1 actions. The manifold M_2 is not stably parallelizable and admits only one free S^1 action.

Proof. This is merely a restatement of the results of theorems 2 and 3.

§3. CLASSIFICATION OF THE INDUCED INVOLUTIONS

Each circle action on a manifold M induces an action of Z_2 on M. We have the following theorem.

THEOREM 5. *Two free differentiable involutions on a 1-connected 6-manifold M with $H_2 M$ free which are the induced involutions of differentiable free circle actions are equivalent if and only if the circle actions are equivalent.*

Proof. This was proved in [2] for the case where the rank of $H_2 M$ is 0. Suppose the rank of $H_2 M$ is a positive integer k. Now by theorem 4 the manifold M must be

$$S^2 \times S^4 \# \cdots \# S^2 \times S^4 \# S^3 \times S^3 \# \cdots \# S^3 \times S^3$$

and it is clear that equivalent circle actions induce equivalent involutions. We must show that the circle action determined by the map h of Theorem 3 from $K \# L \# \cdots \# L$ into CP^∞ and the circle action determined by the map f from $L \# L \# \cdots \# L$ into CP^∞ which is non-trivial on the first factor and trivial on all the others determine distinct involutions. It follows from [2] that the orbit spaces of the induced involutions are

$$G_h = [G - (S^1 \times \text{int } D^5)] \cup [S^1 \times (L \# \cdots \# L) - (S^1 \times \text{int } D^4)]$$

and $G_f = [RP^3 \times S^3 - (S^1 \times \text{int } D^5)] \cup [S^1 \times (L \# \cdots \# L) - (S^1 \times \text{int } D^5)]$, where G is the non-trivial 3-sphere bundle over RP^3, and the identifications on the boundary are obvious. It also follows from [2] that the second Stiefel–Whitney class of G_h is non-zero. But the second Stiefel–Whitney class of G_f is easily seen to be zero since each of $RP^3 \times S^3$ and $S^1 \times (L \# \cdots \# L)$ are parallelizable and the identification will not change ω_2. Hence G_h and G_f are not homotopy equivalent and hence the involutions are not even topologically equivalent.

§4. THE CLASSIFICATION WHEN H_2M IS NOT FREE

Suppose M is a 1-connected 6-manifold and G is a free differentiable action of S^1 on M. As before, the homotopy sequence of the fibration $S^1 \to M \to M/G$ shows that $\pi_1(M/G)$ is trivial. Assume that $f: M/G \to CP^\infty$ is a classifying map for the principle S^1 bundle $S^1 \to M \to M/G$. Then we have the following commutative diagram where the horizontal rows are given by the Gysin sequence of $S^1 \to M \to M/G$ and the universal bundle $S^1 \to S^\infty \to CP^\infty$.

$$\begin{array}{ccccccccc}
\to H_1(M/G) & \to & H_2 M & \to & H_2(M/G) & \to & H_0(M/G) & \to & 0 \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\
\to H_1(P^\infty) & \to & H_2 S^\infty & \to & H_2(CP^\infty) & \to & H_0(CP^\infty) & \to & 0.
\end{array}$$

This diagram reduces to the following diagram

$$\begin{array}{ccccccccc}
0 & \to & H_2 M & \to & H_2(M/G) & \to & Z & \to & 0 \\
 & & \downarrow & & \downarrow f_* & \approx & \downarrow \approx & & \\
0 & \to & 0 & \to & Z & \to & Z & \to & 0.
\end{array}$$

Now there is a splitting homomorphism from Z to $H_2(M/G)$ and a decomposition of $H_2(M/G)$ into $Z \oplus H_2 M$ such that f_* is the identity on the first factor and zero on the second. It is now clear that the torsion subgroup of $H_2 M$ is isomorphic to the torsion subgroup of $H_2(M/G)$, and that f_* maps the torsion subgroup of $H_2(M/G)$ to zero. Again by [1] we know that M/G is diffeomorphic to $N_1^5 \# N_2^5$, where $H_2 N_1$ is free and $H_2 N_2$ has no free generators. Hence the S^1 bundle over N_2 is trivial, and in the first part of the paper we classified all the S^1 actions over N_1, so we know how to classify these actions.

THEOREM 6. *Given a* 1-*connected* 6-*manifold* M, *it will admit a free differentiable* S^1 *action iff* (1) *The torsion subgroup of* $H_2 M$ *is either trivial or the direct sum of groups of the form* $Z_k \oplus Z_k$, *and* (2) M *is equivariantly diffeomorphic to* $M_1 \cup_g M_2$, *where* M_1 *is one of the manifolds of Theorem* 4 *with the interior of a product neighborhood of an orbit removed,* M_2 *is* $S^1 \times (N^5 - \text{int } D^5)$, *where* $H_2 N^5$ *is the torsion subgroup of* $H_2 M$, *and* g *is the obvious identification along the boundary.*

Proof. The only thing to observe is that the restriction on $H_2 M$ is necessary since these are the only groups which can be the torsion groups of a 1-connected 5-manifold.

Unfortunately, the torsion subgroups of a 1-connected 5-manifold do not necessarily determine the manifold. It is necessary to also know the homomorphism on the second homology which defines ω_2 to completely determine the manifold.

Now as in the earlier sections we can determine if a 1-connected 6-manifold does admit a free differentiable S^1 action, and if it does, as before, it can admit at most two such actions. Similarly, distinct S^1 actions will induce distinct involutions.

REFERENCES

1. D. BARDEN: Simply connected five-manifolds, *Ann. Math.* **82** (1965), 365–385.
2. R. GOLDSTEIN and L. LININGER: Actions on 2-connected 6-manifolds, *Am. J. Math.* **91** (1969), 499–504.
3. W. C. HSIANG: A note on free differentiable actions of S^1 and S^3 on homotopy spheres, *Ann. Math.* **83** (1966), 266–272.
4. R. LEE: Piecewise linear classification of some free Z_p actions on S^{4k+3}, to appear in the *Mich. Math. J.*
5. L. LININGER: On topological transformation groups, *Proc. Am. math. Soc.* **20** (1969), 191–192.
6. D. MONTGOMERY and C. T. YANG: Differentiable actions on homotopy seven spheres, *Trans. Am. Math. Soc.* **122** (1966), 480–489.
7. P. MOSTERT: Proc. of the Conf. on Transformation groups, Springer, New York (1968).
8. C. T. C. WALL: Classification problems in differential topology V, *Invent. Math.* **1** (1966), 355–375.

State University of New York at Albany